Section 1: Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”

- Gilbert Strang, *Linear Algebra and its Applications*
Vectors

\[ \mathbf{\bar{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} \]

In two or three dimensions, we can draw these as arrows:

N=2

In higher dimensions, we typically must resort to a “spike-plot”:

N=15

Vector operations

• scalar multiplication
• addition, vector spaces
• length, unit vectors
• inner product (a.k.a. “dot” product)
  – properties: commutative, distributive
  – geometry: cosines, orthogonality test

[on board: geometry]
Vectors as “operators”

- “averager”
- “windowed averager”
- “gaussian averager”
- “local differencer”
- “component selector”

Inner product with a unit vector

- projection
- distance
- change of coordinates
**Linear System**

A system $S$ is a linear system if (and only if) it obeys the principal of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For any input vectors $\{\vec{x}, \vec{y}\}$, and any scalars $\{a, b\}$, the two diagrams at the right must produce the same response:

**Linear Systems**

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models
Implications of Linearity

Input
\[ \vec{v} \]

Output

"impulse" vectors
"standard basis"
"axis vectors"
Implications of Linearity

Response to \textit{any} input can be predicted from responses to impulses. This defines the operation of \textit{matrix multiplication}.

Matrix multiplication

- input perspective: weighted sum of columns
  (from diagram on previous slide)
- output perspective: inner product with rows
- distributive property (directly from linearity)
- associative property - cascade of two linear systems defines the product of two matrices
- generally \textit{not} commutative ($AB \neq BA$),
  but note that $(AB)^T = B^T A^T$
Matrix multiplication: dimensional consistency

Orthogonal matrices
- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Diagonal matrices
- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square: creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

Identity matrix

Matrix types
Singular Value Decomposition (SVD)

- \( M = U S V^T \), "rotate, stretch, rotate"
- \( V \) is input coordinate system (\( U \), output)
- interpretation as sum of outer products
- non-uniqueness (permutations, sign flips)
- nullspace and rangespace
- inverse and pseudo-inverse

\[ \text{[details on board]} \]
\[ M \bar{x} = \sum_{k} s_k \begin{pmatrix} \bar{u}_k^T \end{pmatrix} \bar{u}_k \]

- **Orthogonal basis for output space**
- **Orthogonal basis for input space**
- **“Singular values”**
- **Orthogonal basis for “range space”**
- **Orthogonal basis for “null space”**
$$M = U S V^T$$

- **$M$**: Orthogonal basis for “range space”
- **$U$**: Orthogonal basis for “null space” (all zeros)
- **$S$**: Diagonal matrix
- **$V^T$**: Transpose of orthogonal matrix

Orthogonal basis for “range space”